Announcements

1) Quiz next week on Thursday, Exam week following on Thursday

2) HW #3 due today

Continuity

If $f: D \to \mathbb{R}$

where D is a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ (or $\mathbb{R}^n$),

we say f is continuous at (a,b) or (a,b,c) in D if

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)} \quad (D \text{ in } \mathbb{R}^2)$$

or

$$\boxed{\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)} \quad (D \text{ in } \mathbb{R}^3)$$

The Chain Rule (part 1)

Section 14.5

Remember one-variable case:

If $y = f(x)$ and $x = x(t)$,

then

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}}$$

provided all derivatives exist!

Example 1: Suppose

$$f(x,y) = x^2 y$$

and $x(t) = \cos(t)$, $y(t) = e^t$.

Then we can compose to get a function

$$g: \mathbb{R} \to \mathbb{R},$$

$$g(t) = f(\cos(t), e^t)$$
$$= \cos^2(t)\, e^t$$

Then

$$\boxed{g'(t) = -2\cos(t)\sin(t)e^t + \cos^2(t)e^t}$$

Find in a different way:

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2$$

$$x'(t) = -\sin(t)$$

$$y'(t) = e^t$$

Compute

$$\frac{\partial f}{\partial x}(x(t), y(t)) \cdot \frac{dx}{dt}$$

$$+ \frac{\partial f}{\partial y}(x(t), y(t)) \cdot \frac{dy}{dt}$$

$$= 2\cos(t)e^t(-\sin(t))$$

$$+ \cos^2(t)e^t$$

$$= -2\cos(t)\sin(t)e^t$$

$$+ \cos^2(t)e^t$$

$$= \boxed{g'(t)}$$

Chain Rule (baby version)

If $z = f(x, y)$ is a function from D in $\mathbb{R}^2$ to $\mathbb{R}$ and $x = x(t)$, $y = y(t)$, then

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

If $w = f(x,y,z)$ and

$x = x(t)$, $y = y(t)$, $z = z(t)$,

$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}}$$

A similar formula holds for $g = f(x_1, x_2, \dots, x_n)$ where $x_i = x_i(t)$ for all $1 \le i \le n$

($f : D \to \mathbb{R}$, D in $\mathbb{R}^n$)

Consequence: (implicit slopes)

Given an equation like $x^2+y^2=1$, the graph is not always the graph of a function $h:\mathbb{R}\to\mathbb{R}$.

However, the graph is a level curve of the function $f(x,y)=x^2+y^2$.

Given an equation of the form $f(x,y) = c$, under certain conditions on f, the implicit function theorem says we can solve for y implicitly as a function of x (on small enough open disks of the form $(x-a)^2 + (y-b)^2 < r$)

Regard y as a function of x, consider

$$g(x) = f(x, y(x)) = C.$$

Take the derivative of both sides

$$0 = \frac{d}{dx}(c) = \frac{d}{dx}(g(x))$$

$$= \frac{\partial f}{\partial x}\overbrace{\frac{dx}{dx}}^{=1} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

(by baby chain rule)

We get

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

and so if $\frac{\partial f}{\partial y} \neq 0$,

$$\boxed{\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}}$$

Example 2: If

$$\tan(xy^3) = 2,$$

find $\dfrac{dy}{dx}$.

We use $\dfrac{dy}{dx} = -\dfrac{\left(\dfrac{\partial f}{\partial x}\right)}{\left(\dfrac{\partial f}{\partial y}\right)}$

$$f(x,y) = \tan(xy^3)$$

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$$\frac{\partial f}{\partial x} = \sec^2(xy^3)\cdot y^3$$

$$\frac{\partial f}{\partial y} = \sec^2(xy^3)(3y^2x)$$

$$\frac{dy}{dx} = -\frac{\cancel{\sec^2(xy^3)}\,y^3}{\cancel{\sec^2(xy^3)}\,(3y^2x)}$$

$$= \frac{-y^3}{3y^2x} = \boxed{-\frac{y}{3x}}$$

Example 2: Suppose

$$f(x,y) = x^3 + \sqrt{y}$$

and now instead of functions of one variable,

$$x = x(s,t) = \ln(s+t)$$

$$y = y(s,t) = s^4 t^8.$$

We can make

$$g(s,t) = f(x(s,t), y(s,t)),$$

what are $\frac{\partial g}{\partial s}$, $\frac{\partial g}{\partial t}$?

$$g(s,t) = (\ln(s+t))^3 + \sqrt{s^4 t^8}$$

$$= (\ln(s+t))^3 + s^2 t^4$$

$$\boxed{\begin{aligned} \frac{\partial g}{\partial s} &= \frac{3(\ln(s+t))^2}{s+t} + 2st^4 \\ \frac{\partial g}{\partial t} &= \frac{3(\ln(s+t))^2}{s+t} + 4s^2t^3 \end{aligned}}$$